

LP decoding excess over symmetric channels

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Abstract

We consider the problem of Linear Programming (LP) decoding of binary linear codes. The LP excess lemma was introduced by the first author, B. Ghazi, and R. Urbanke (IEEE Trans. Inf. Th., 2014) as a technique to trade crossover probability for “LP excess” over the Binary Symmetric Channel. We generalize the LP excess lemma to discrete, binary-input, Memoryless, Symmetric and LLR-Bounded (MSB) channels. As an application, we extend a result by the first author and H. Audah (IEEE Trans. Inf. Th., 2015) on the impact of redundant checks on LP decoding to discrete MSB channels.

1 Introduction

In 2003, Feldman [1] introduced Linear Programming (LP) decoding as a relaxation of Maximum Likelihood (ML) decoding. The good performance of LP decoding of LDPC codes and its relation to iterative decoding was established in multiple studies such as [2, 3, 4, 5] (a comprehensive survey is found in [7]).

The LP excess lemma was introduced and established in [6] in the context of the Binary Symmetric Channel (BSC) as a technique to trade crossover probability for “LP excess” when analyzing the LP decoder error probability under the assumption that the all zeros codeword was transmitted. The lemma says that if the LP decoder works on a slightly noisier channel, we can guarantee that it corrects a slightly shifted-down version of the received LLRs. In dual terms, this implies the existence of a dual witness [4] where the variable nodes inequalities are satisfied on the variable nodes with some constant positive “LP excess”. The lemma was used to study the LP decoding thresholds of spatially coupled codes [6] and the impact of redundant parity checks on the LP decoding thresholds of LDPC codes on the BSC [7].

In this paper we extend the LP excess lemma from the BSC to discrete, binary-input, Memoryless, Symmetric and LLR-Bounded (MSB) channels. We define the channel model in Section 1.1 and we give the needed background on LP decoding in Section 1.2. We state and prove our main result in Section 2. As an application, we use the extended lemma in Section 3 to extend the result of [7] to discrete MSB channels.

1.1 Channel model

We consider MSB channels: an *MSB channel* [3] is a *binary-input Memoryless* channel where the input alphabet is $\{0, 1\}$ and the transition probability has a *Symmetry* property as well as a *Bounded LLR* property. For simplicity of the presentation, we assume that the channel is *discrete*, i.e., the output alphabet Σ is a finite set (or a countably infinite set). The channel is *symmetric* in the sense that we have a partition of Σ into pairs (a, a^*) , such that $\Pr(a|0) = \Pr(a^*|1)$ and $\Pr(a|1) = \Pr(a^*|0)$. The *pairing* is a bijective map

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$*$: $\Sigma \rightarrow \Sigma$ such that $a^{**} = a$ for each $a \in \Sigma$. Thus the channel is fully specified by a triplet $ch = (\Sigma, p, *)$, where p is a probability distribution on Σ when 0 was transmitted, i.e., $\Pr(a|0) = p(a)$ and $\Pr(a|1) = p(a^*)$. The Log-Likelihood-Ratios (LLR) $L_{ch}(\cdot) = L(\cdot)$ is a real-valued map on Σ given by

$$L(a) = \ln \frac{p(a)}{p(a^*)}.$$

Note that $L(a) = -L(a^*)$ for each $a \in \Sigma$. We assume that the channel is *LLR bounded* in the sense that $\|L\|_\infty$ is upper bounded by a constant. If Σ is finite, LLR boundedness is equivalent to $p(a) \neq 0$ for all $a \in \Sigma$. We denote by $\mu_{ch} = \mu$ the LLR probability distribution given 0 is transmitted, i.e., μ is the probability distribution of $L(a)$ where a is sampled according to p .

The importance of discrete MSB channels stems from the fact that they allow the decoder to use soft quantized information. They include for example the BSC, the mixed BSC-erasure channel and the finitely-quantized additive Gaussian-noise channel. The binary erasure channel is an example of a discrete symmetric channel with possibly infinite LLRs.

We are interested in small *distortions* of discrete MSB channels:

Definition 1.1 (Channel distortion). *If $ch = (\Sigma, p, *)$ is a discrete MSB channel and $\alpha > 0$, we call channel ch' an α -distortion of ch if $ch' = (\Sigma, p', *)$ for some probability distribution p' on Σ such that the L_1 -distance*

$$\|p - p'\|_1 := \sum_a |p(a) - p'(a)| \leq \alpha.$$

Note that, ch' shares with ch the same paring map $$.*

For instance, consider the β -BSC channel with cross over probability β . An α -distortion of the β -BSC is the β' -BSC where $|\beta - \beta'| \leq \alpha/2$.

Notations. In this document we use a bold-faced notation to refer to n -dimensional vector: we transmit a length- n binary string $\mathbf{x} \in \{0, 1\}^n$ and receive $\mathbf{y} \in \Sigma^n$ of \mathbf{x} . Additionally, we denote by p^n the product distribution on Σ^n associated with p and μ^n the product distribution on \mathbb{R}^n associated with μ . Thus, if $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is the all-zeros vector, then \mathbf{y} is distributed according to p^n and the corresponding LLR vector

$$\boldsymbol{\gamma} = \mathbf{L}(\mathbf{y}) := (L(y_i))_{i=1}^n \in \mathbb{R}^n$$

is distributed according to μ^n .

1.2 LP decoding

Let $Q \subset \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code with blocklength n and $ch = (\Sigma, p, *)$ a discrete MSB channel. Consider transmitting a codeword $\mathbf{x} \in Q$ over ch , which outputs $\mathbf{y} \in \Sigma^n$. The ML decoder of Q is given by

$$\mathbf{ML}(\mathbf{y}) = \arg \max_{\mathbf{x} \in Q} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}).$$

In terms of the LLR vector $\boldsymbol{\gamma} = \mathbf{L}(\mathbf{y})$, the ML decoder is given by

$$\mathbf{ML}_Q(\boldsymbol{\gamma}) = \arg \min_{\mathbf{x} \in Q} \langle \mathbf{x}, \boldsymbol{\gamma} \rangle,$$

where $\langle \mathbf{x}, \boldsymbol{\gamma} \rangle := \sum_i x_i \gamma_i$.

Feldman *et al.* [1, 2] introduced the notion of LP decoding, which is based on relaxing the optimization problem on Q into a LP. Due to the linearity of the objective function $\langle \mathbf{x}, \boldsymbol{\gamma} \rangle$, optimizing over Q is equivalent

to optimizing over the convex polytope $\text{conv}(Q) \subset \mathbb{R}^n$ spanned by the convex combinations of the codewords in Q . The idea of Feldman is to relax $\text{conv}(Q)$ into a larger lower-complexity polytope. In general terms, an *LP-relaxation* of Q is a Q -symmetric convex polytope $P \subset [0, 1]^n$, where *Q -symmetry* means that $(|\mathbf{x}_i - \mathbf{y}_i|)_{i=1}^n \in Q$, for each $\mathbf{x} \in Q$ and $\mathbf{y} \in P$ [1]. Note that Q -symmetry implies that $Q \subset P$.

The *LP decoder* is given by

$$\mathbf{LP}_P(\gamma) = \arg \min_{\mathbf{x} \in P} \langle \mathbf{x}, \gamma \rangle.$$

While useful constructions of P are obtained from Tanner graph representations [1, 2], it is simpler to establish the LP-excess lemma in the general framework of Q -symmetric polytopes $P \subset [0, 1]^n$. The Q -symmetry of P implies that when evaluating the LP decoding error probability, we can assume without loss of generality that the all-zeros codeword $\mathbf{0}$ was transmitted [1]. Thus $\gamma \sim \mu^n$, where $\mu = \mu_{ch}$ is the LLR probability distribution given 0. As in previous works [1, 2], we assume that the LP decoder fails if $\mathbf{0}$ is not the unique optimal solution of the LP, i.e., the P -LP decoder *succeeds* on γ iff $\mathbf{LP}_P(\gamma) = \mathbf{0}$.

We say that the LP decoder *succeeds with LP excess* ξ on γ if it succeeds on $\gamma - \xi \mathbf{1}$, i.e., $\mathbf{LP}_P(\gamma - \xi \mathbf{1}) = \mathbf{0}$, where $\mathbf{1} \in \mathbb{R}^n$ is the all ones vector and $(\gamma - \xi \mathbf{1})_i = \gamma_i - \xi$, for $i = 1, \dots, n$.

For constructions of P from a Tanner graphs, LP excess can be interpreted in terms of the notion of a dual-witness [4] as follows. In dual terms, the P -LP decoder succeeds with LP excess ξ on γ iff $\gamma - \xi \mathbf{1}$ has a dual-witness, i.e., γ has a dual-witness where each of the dual-witness variable nodes inequalities is satisfied with “LP excess ξ ” (see Definition 2.1 and Theorem 2.2 in [7] for the equivalent dual characterizations of LP decoding success).

When studying the LP decoding error probability as the block length n tends to infinity, we consider an *infinite family* of \mathbb{F}_2 -linear codes $\mathcal{Q} = \{Q_n\}_n$ and an associated infinite family of LP-relaxation $\mathcal{P} = \{P_n\}_n$. We say that the \mathcal{P} -LP decoder succeeds on *ch* with high probability if

$$\lim_{n \rightarrow \infty} \Pr_{\gamma \sim \mu^n} [\mathbf{LP}_{P_n}(\gamma) \neq \mathbf{0}] = 0.$$

We say that the \mathcal{P} -LP decoder “*succeeds on ch with LP excess ξ with high probability*” if

$$\lim_{n \rightarrow \infty} \Pr_{\gamma \sim \mu^n} [\mathbf{LP}_{P_n}(\gamma - \xi \mathbf{1}) \neq \mathbf{0}] = 0.$$

2 LP excess lemma

In this section, we extend the BSC LP excess lemma [6] stated below to discrete MSB channels.

Lemma 2.1 ([6]). (BSC LP Excess Lemma: trading crossover probability with LP excess) *Consider the β -BSC which crossover probability $0 < \beta < 1/2$. Let \mathcal{Q} be an infinite family of \mathbb{F}_2 -linear codes and \mathcal{P} an associated family of LP-relaxations.*

Assume that there exists $\beta < \beta' < 1/2$ such that the \mathcal{P} -LP decoder succeeds on the β' -BSC with high probability.

Then, there exists a $\xi > 0$ such that the \mathcal{P} -LP decoder succeeds on the β -BSC with LP excess ξ with high probability.

Lemma 2.2. (MSB LP Excess Lemma: trading channel distortion with LP excess) *Let ch be a discrete MSB channel, \mathcal{Q} an infinite family of \mathbb{F}_2 -linear codes and \mathcal{P} an associated family of LP-relaxations.*

Assume that there exists $\alpha > 0$ such that for each α -distortion ch' of ch , the \mathcal{P} -LP decoder succeeds on ch' with high probability.

Then, there exists $\xi > 0$ such that the \mathcal{P} -LP decoder succeeds on ch with LP excess ξ with high probability.

In proving the Lemma, we follow similar steps to those taken in [6]. The starting point in [6] is to realize the β' -BSC as a distortion of the β -BSC resulting from the bit-wise OR of the β -BSC error event with an independent Bernoulli random variable B . The distorted channel operates according to the original channel if $B = 0$ and it produces an error if $B = 1$. To generalize this construction, we use a similar Bernoulli-induced distortion of ch . The key new ingredient is a construction of a probability distribution q supported on the set of output symbols with negative LLRs. The distorted channel ch' operates according to the original channel if $B = 0$ and according to q if $B = 1$. A key property of the constructed q will be that the LLR map L' of ch' is a positive constant scale of that of ch , i.e., there exists a constant $c \in (0, 1)$ such that $L'(a) = cL(a)$ for all $a \in \Sigma$. This property will be essential in extending the argument of [6] to our setup.

Proof of Lemma 2.2. The proof is based on the fundamental cone. Let $C_n \subset \mathbb{R}^n$ be the fundamental cone [5] of the P_n -LP decoder, i.e., the set of all LLR vectors correctly decoded by the decoder:

$$C_n = \{\gamma \in \mathbb{R}^n : \mathbf{LP}_{P_n}(\gamma) = \mathbf{0}\}.$$

Since $\mathbf{LP}_{P_n}(\gamma - \xi \mathbf{1}) = \mathbf{0}$ is equivalent to $\gamma \in C_n + \xi \mathbf{1}$, our objective is to show that there exists a $\xi > 0$ such that $\mu^n(C_n + \xi \mathbf{1}) = 1 - o_n(1)$. The hypothesis of the theorem guarantees that for any α -distortion ch' of ch , $\mu^n(C_n) = 1 - o_n(1)$, where $\mu' = \mu_{ch'}$ is the LLR probability distribution of ch' given 0.

By the definition of the LP decoder, C_n is the interior of the polar cone of P_n , i.e.,

$$C_n = \{\gamma \in \mathbb{R}^n : \langle \gamma, \mathbf{x} \rangle > 0 \text{ for each nonzero } \mathbf{x} \in P_n\}.$$

We note that since $P_n \subset [0, 1]^n \subset (\mathbb{R}^+)^n$, C_n is closed under translation by vectors in the non-negative quadrant, i.e., $C_n + (\mathbb{R}^+)^n \subset C_n$. We will argue that ξ exists using only the property that $C_n \subset \mathbb{R}^n$ is a convex cone such that $C_n + (\mathbb{R}^+)^n \subset C_n$.

Consider the partition of Σ into three sets:

$$\begin{aligned} \Sigma_- &= \{a \in \Sigma : p(a) < p(a^*)\} \\ \Sigma_0 &= \{a \in \Sigma : p(a) = p(a^*)\} \\ \Sigma_+ &= \Sigma_-^*. \end{aligned}$$

Thus L is negative on Σ_- , zero on Σ_0 and positive on Σ_+ . Without loss of generality, we assume that Σ_- and Σ_+ are nonempty (otherwise, the channel capacity is zero).

Let $0 < \delta < 1$ be a constant such that $\delta \leq \alpha/2$ and define channel $ch' = (\Sigma, p', *)$, where p' is the distribution on Σ given by

$$\begin{aligned} p'(a) &= \delta q(a) + (1 - \delta)p(a) & \text{if } a \in \Sigma_- \\ p'(a) &= (1 - \delta)p(a) & \text{if } a \in \Sigma_0 \cup \Sigma_+, \end{aligned}$$

and where q is a probability distribution on Σ_- that will be specified later. We will sample from p' as follows. First we sample a Bernoulli random variable $B \sim \text{Ber}(\delta)$ which takes the value 1 with probability δ . If $B = 0$, we sample from p and if $B = 1$, we sample from q . Channel ch' is an α -distortion of ch because $\|p - p'\|_1 \leq 2\delta \leq \alpha$. The LLR map of ch' denoted by L' is given by:

$$L'(a) = \ln \left[\frac{\delta q(a) + (1 - \delta)p(a)}{(1 - \delta)p(a^*)} \right] \text{ if } a \in \Sigma_-,$$

$L'(a) = -L'(a^*)$ if $a \in \Sigma_+$ and $L'(a) = 0$ if $a \in \Sigma_0$. We choose q so that there exists a constant $c \in (0, 1)$ such that $L'(a) = cL(a)$ for all $a \in \Sigma$ which is guaranteed by enforcing $L'(a) = cL(a)$ on $a \in \Sigma_-$, i.e.,

$$\frac{\delta q(a) + (1 - \delta)p(a)}{(1 - \delta)p(a^*)} = \left(\frac{p(a)}{p(a^*)} \right)^c, \quad a \in \Sigma_-.$$

Solving for $q(\cdot)$, we get

$$q(a) = \frac{1 - \delta}{\delta} p(a) \left[\left(\frac{p(a^*)}{p(a)} \right)^{1-c} - 1 \right], \quad a \in \Sigma_-.$$

Since $c \in (0, 1)$ and $p(a^*) > p(a)$ for $a \in \Sigma_-$, we have $q(a) > 0$ on $a \in \Sigma_-$. To guarantee that $\sum_a q(a) = 1$, we choose $c \in (0, 1)$ so that $s(c) = \frac{\delta}{1-\delta}$, where

$$s(c) := \sum_{a \in \Sigma_-} p(a) \left[\left(\frac{p(a^*)}{p(a)} \right)^{1-c} - 1 \right].$$

This follows from the continuity of $s(\cdot)$ as a function of c and the facts that $s(1) = 0$ and

$$s(0) = \sum_{a \in \Sigma_-} p(a^*) - \sum_{a \in \Sigma_-} p(a) = p(\Sigma_+) - p(\Sigma_-) > 0,$$

since Σ_+ and Σ_- are assumed to be nonempty. In what follows, fix $\delta \in (0, 1)$ to be any constant such that $\delta \leq \frac{\alpha}{2}$ such that $\frac{\delta}{1-\delta} < p(\Sigma_+) - p(\Sigma_-)$ to guarantee the existence of q and c .

In the remainder of the proof we follow the steps in [6]: we use an averaging argument followed by Markov Inequality. For clarity, we will use capital letters to refer to random quantities. Define $\mathbf{f} : \Sigma^n \times \Sigma^n \times \{0, 1\}^n \rightarrow \Sigma^n$ by

$$\mathbf{f}(\mathbf{y}, \mathbf{z}; \mathbf{b})_i = \begin{cases} \mathbf{z}_i & \text{if } \mathbf{b}_i = 1 \\ \mathbf{y}_i & \text{if } \mathbf{b}_i = 0. \end{cases}$$

Thus, if $\mathbf{Y} \sim p^n$, $\mathbf{Z} \sim q^n$ and $\mathbf{B} \sim \text{Ber}(\delta)^n$, then $\mathbf{f}(\mathbf{Y}, \mathbf{Z}; \mathbf{B})$ is distributed according to p^n , and $\gamma' = \mathbf{L}'(\mathbf{f}(\mathbf{Y}, \mathbf{Z}; \mathbf{B}))$ is according to μ'^n . For each $\mathbf{y} \in \Sigma^n$, define the random vector

$$\mathbf{\Gamma}'(\mathbf{y}, \mathbf{Z}; \mathbf{B}) = \beta \mathbf{L}'(\mathbf{f}(\mathbf{y}, \mathbf{Z}; \mathbf{B})) = \beta c \mathbf{L}(\mathbf{f}(\mathbf{y}, \mathbf{Z}; \mathbf{B})) \in \mathbb{R}^n$$

over the random choice of $\mathbf{Z} \sim q^n$ and $\mathbf{B} \sim \text{Ber}(\delta)^n$, where $\beta > 0$ is a constant to be specified later. Denoting by $1_{C_n} : \mathbb{R}^n \rightarrow \{0, 1\}$ the indicator function of C_n (i.e. $1_{C_n}(\gamma) = 1$ iff $\gamma \in C_n$), we define $\mathbf{w}(\mathbf{y}) \in \mathbb{R}^n$ for $\mathbf{y} \in \Sigma^n$ by

$$\mathbf{w}(\mathbf{y}) = \mathbb{E}_{\mathbf{Z}, \mathbf{B}} [\mathbf{\Gamma}'(\mathbf{y}, \mathbf{Z}; \mathbf{B}) \times 1_{C_n}(\mathbf{\Gamma}'(\mathbf{y}, \mathbf{Z}; \mathbf{B}))]. \quad (1)$$

For each $\mathbf{y} \in \Sigma^n$ we have $\mathbf{w}(\mathbf{y}) \in C_n$ since C_n is a convex cone. Thus, interpreting vector inequalities coordinate-wise,

$$\mu^n(C_n + \xi \mathbf{1}) \geq \Pr_{\mathbf{Y} \sim p^n} [(\mathbf{L}(\mathbf{Y}) - \mathbf{w}(\mathbf{Y})) \geq \xi \mathbf{1}] \quad (2)$$

because $\mathbf{v} \geq \mathbf{w}(\mathbf{y})$, for any $\mathbf{v} \in C_n$ and any $\mathbf{y} \in \Sigma^n$ since $C_n + (\mathbb{R}^+)^n \subset C_n$. Equation (1) can be written as

$$\mathbf{w}(\mathbf{y}) = \mathbb{E} [\mathbf{\Gamma}'(\mathbf{y}, \mathbf{Z}; \mathbf{B})] - \mathbb{E} [\mathbf{\Gamma}'(\mathbf{y}, \mathbf{Z}; \mathbf{B}) | \varphi(\mathbf{y}, \mathbf{Z}; \mathbf{B})] \cdot \Phi(\mathbf{y}),$$

where $\varphi(\mathbf{y}, \mathbf{Z}, \mathbf{B})$ is the error event “ $\mathbf{T}'(\mathbf{y}, \mathbf{Z}; \mathbf{B}) \notin C_n$ ” and

$$\Phi(\mathbf{y}) := \Pr_{\mathbf{Z}, \mathbf{B}}[\varphi(\mathbf{y}; \mathbf{Z}, \mathbf{B})].$$

The first term

$$\mathbb{E} [\mathbf{T}'(\mathbf{y}, \mathbf{Z}; \mathbf{B})_i] = \beta c(1 - \delta)L(\mathbf{y}_i) - \beta c\delta s,$$

where

$$s := -\mathbb{E}_{Z \sim q}[L(Z)]$$

is a positive scalar because q is supported on Σ_- and $\mathbb{E}_{Z \sim q}[L(Z)]$ is strictly negative. The second term

$$\mathbb{E} [\mathbf{T}'(\mathbf{y}, \mathbf{Z}; \mathbf{B})_i | \varphi(\mathbf{y}, \mathbf{Z}; \mathbf{B})] \geq -\beta \|L'\|_\infty = -\beta c \|L\|_\infty$$

since the LLRs are bounded. It follows that

$$\mathbf{w}(\mathbf{y})_i \leq \beta c[(1 - \delta)L(\mathbf{y}_i) - \delta s + \|L\|_\infty \Phi(\mathbf{y})].$$

Setting $\beta = \frac{1}{c(1-\delta)}$, we get

$$\mathbf{w}(\mathbf{y})_i \leq L(\mathbf{y}_i) - \frac{\delta s - \|L\|_\infty \Phi(\mathbf{y})}{1 - \delta}.$$

Therefore, to guarantee that the vector inequality $\mathbf{L}(\mathbf{y}) - \mathbf{w}(\mathbf{y}) \geq \xi \mathbf{1}$ holds, it is enough to require the scalar inequality $\delta s - \|L\|_\infty \Phi(\mathbf{y}) \geq \xi(1 - \delta)$. Note this reduction of the vector inequality to a scalar inequality critically depends on the choice of q so that $L' = cL$. Setting $\xi = \frac{\delta s}{2(1-\delta)}$, we get from (2) that

$$1 - \mu^n(C_n + \xi \mathbf{1}) \leq \Pr_{\mathbf{Y}} \left[\Phi(\mathbf{Y}) > \frac{\delta s}{2\|L\|_\infty} \right].$$

Using Markov Inequality, and the fact that $\mathbb{E}_{\mathbf{Y}} [\Phi(\mathbf{Y})] = 1 - \mu^n(C_n)$, we obtain

$$1 - \mu^n(C_n + \xi \mathbf{1}) \leq \frac{2\|L\|_\infty}{\delta s} (1 - \mu^n(C_n)).$$

Since $\mu^n(C_n) = 1 - o_n(1)$, we conclude that $\mu^n(C_n + \xi \mathbf{1}) = 1 - o_n(1)$, where $\xi > 0$ is constant which depends on α and the channel ch . \square

Remark 2.3. I) If we replace probability distributions with densities, the LP excess lemma and its proof hold for continuous MSB channels.

II) We conjecture that the LLR boundedness is not needed for the lemma to hold. One justification of this conjecture is the Gaussian channel discussed below.

2.1 Gaussian channel

On the σ -Additive White Gaussian Noise (σ -AWGN) channel, we receive $Y = (-1)^x + \sigma Z$, where $x = 0$ or 1 is the transmitted bit and $Z \sim \mathcal{N}(0, 1)$, the standard Gaussian distribution. The AWGN has unbounded LLRs.

By a simple scaling argument, the following version of the LP excess lemma holds on the AWGN:

Lemma 2.4. Let $Q_n \subset \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code, $P_n \subset \mathbb{R}^n$ an LP-relaxation of Q_n and $\sigma' > \sigma > 0$. The probability of success of the P_n -LP decoder on the σ' -AWGN is equal to its probability of success on the σ -AWGN with LP excess ξ , where $\xi = \frac{\sigma' - \sigma}{\sigma'}$.

Proof. The LLR map is $L(y) = \frac{2}{\sigma^2}y$ (e.g., [1]). Assume that $\mathbf{0}$ was transmitted and let μ and μ' be the LLR densities associated with σ and σ' , respectively. Since

$$\frac{\sigma}{\sigma'}(1 + \sigma'z) = 1 + \sigma z - \xi,$$

we get $\mu^n(C_n) = \mu^n(C_n + \xi \mathbf{1})$, for each $C_n \subset \mathbb{R}^n$ closed under multiplication by positive scalars and in particular for the fundamental cone C_n of the P_n -LP decoder. \square

The distinguishing features of the AWGN from other channels in this context are: (1) scaling Z corresponds to distorting the channel and (2) the LLR map is linear in y .

3 Application to redundant parity checks

The BSC LP excess lemma was used in [7] to show that the LP decoding threshold of LDPC codes on the BSC remains the same upon adding all redundant parity checks, assuming that the underlying Tanner graph has *bounded degree* and possesses two natural properties called *asymptotic strength* and *rigidity* (see Corollary 1.7 in [7]). One implication of this result is that the BSC threshold is a function of the dual code and is not tied to the particular Tanner graph realization of the code. We use in this section our extension of the LP excess lemma to extend the result of [7] from the BSC to discrete MSB channels:

Theorem 3.1. *Let $\mathcal{G} = \{G_n\}_n$ be an infinite family of Tanner graphs, where G_n has n variable nodes. Let $\bar{\mathcal{G}} = \{\bar{G}_n\}_n$ be the resulting family of Tanner graphs obtained by adding all redundant checks, i.e., the parity check nodes of \bar{G}_n correspond to all the nonzero elements of the dual code of G_n . Assume that \mathcal{G} has bounded check degree and that \mathcal{G} is asymptotically strong and rigid. Let ch be a discrete MSB channel. Assume that there exists $\alpha > 0$ such that for each α -distortion ch' of ch , the $\bar{\mathcal{G}}$ -LP decoder succeeds on ch' with high probability. Then, the \mathcal{G} -LP decoder succeeds on ch with high probability.*

In order to prove the theorem we only need the following extension of Theorem 1.2 in [7] to discrete MSB channels:

Lemma 3.2. *Let $\mathcal{G}, \bar{\mathcal{G}}, ch, \alpha, ch'$ be as in Theorem 3.1, and Let d be the maximum degree of a check node in \mathcal{G} . For $k \geq d$, let $\bar{\mathcal{G}}^k := \{\bar{G}_n^k\}_n$ be the resulting family of Tanner graphs obtained by including all redundant checks of degree at most k . There exists a sufficiently large constant $k \geq d$ —where k depends on α and the channel only—such that the $\bar{\mathcal{G}}^k$ -LP decoder succeeds on ch with high probability.*

Proof of Theorem 3.1. Following the proof of Corollary 1.7 in [7], Theorem 3.1 follows from Lemma 3.2 and the rigidity of \mathcal{G} which implies that for each constant $k \geq d$, the LP decoding polytope $P(\bar{G}_n^k) = P(G_n)$ for n large enough. \square

Proof of Lemma 3.2. We use below the terminology of the proof Theorem 1.2 in [7] to explain the needed modifications. At a high level, the following changes are needed:

- Instead of a variable received correctly or in error, we have positive or nonpositive LLRs respectively.
- The value of LP excess is ξ instead of $\frac{\delta}{4}$.
- The maximum absolute value of a received LLR is the *constant* $\|L\|_\infty$ instead of 1.

More specifically, consider operating the \overline{G}_n -LP decoder on ch : assume that the all-zeros codeword was transmitted and consider the received LLR vector $\gamma \sim \mu_{ch}^n$. By the LP excess lemma, there exists a constant $\xi > 0$ (dependent on α) such that with high probability, the \overline{G}_n -LP decoder corrects γ with LP excess ξ , i.e., it corrects $\gamma - \xi \mathbf{1}$. In what follows, consider any such $\gamma \in \mathbb{R}^n$. To verify Lemma 3.2, we will show that the \overline{G}_n^k -LP decoder corrects γ for a sufficiently large constant $k \geq d$ which depends on ξ and the channel (and does not depend on n). For notational simplicity, we will denote G_n, \overline{G}_n^k and \overline{G}_n by G, \overline{G}^k and \overline{G} , respectively. Also, let E, \overline{E}^k and \overline{E} be the set of edges of G, \overline{G}^k and \overline{G} , respectively.

By Theorem 2.2 in [7], there is a hyperflow $w : \overline{E} \rightarrow \mathbb{R}$ in \overline{G} for $\gamma - \xi \mathbf{1}$. Hence,

$$\mathbf{F}(w) < \gamma - \xi \mathbf{1},$$

where $\mathbf{F}(w) \in \mathbb{R}^n$ is the flow as specified in Definition 2.1 in [7]. Let

$$V^+ = \{i : \gamma_i - \xi > 0\}$$

and

$$V^- = \{i : \gamma_i - \xi \leq 0\}$$

be the set of variables nodes with positive and nonpositive “shifted LLR” respectively. Since \overline{G} contains all redundant checks, we can assume by Lemma 4.2 in [7] that w is primitive, hence the inflow to each variable in V^+ is zero and the outflow from each variable in V^- is zero. Following [7], define the *trimmed hyperflow* and the resulting *risky* and *problematic* variables as follows. Trim w by removing all check nodes of degree larger than k . The trimming process leads to a distorted dual witness $w^k : \overline{E}^k \rightarrow \mathbb{R}$ in \overline{G}^k . The *problematic* variables nodes are those for which the hyperflow variables nodes inequalities of w^k are violated with respect to γ . A variable node is called *risky* if it receives at least $\frac{\xi}{2}$ flow from the removed check nodes, thus all the problematic variables are risky. The set of risky variable nodes is called U . We have $U \subset V^-$ since w is primitive. Hence

$$\begin{aligned} \mathbf{F}_i(w^k) &\leq 0 && \text{if } i \in U, \text{ and} \\ \mathbf{F}_i(w^k) &< \gamma_i - \xi/2 && \text{if } i \notin U. \end{aligned}$$

Since all the removed checks have degree larger than k and since $\gamma_i \leq \|L\|_\infty$ for each i , the removed checks give the variables in V^- at most

$$\frac{|V^+| \cdot (\|L\|_\infty - \xi)}{k-1} \leq \frac{n\|L\|_\infty}{k-1}$$

flow. It follows that

$$|U| \leq \frac{2n\|L\|_\infty}{\xi(k-1)}.$$

Since w is primitive, to fix w^k on the problematic variables, it is enough to give each variable in U an $\|L\|_\infty$ flow. Following [7], we do that by exploiting the asymptotic strength of \mathcal{G} and the remaining excess on the nonrisky variable nodes. The remaining LP excess on each nonrisky variable is at least $\xi - \frac{\xi}{2} = \frac{\xi}{2}$. Consider the asymmetric LLR vector $\tau \in \mathbb{R}^n$ given by:

$$\tau_i = \begin{cases} -\|L\|_\infty & \text{if } i \in U \\ \frac{\xi}{2} & \text{otherwise.} \end{cases}$$

We use the remaining excess to fix w^k by superposing w^k with a dual witness for τ .

Since \mathcal{G} is *asymptotically strong*, there exists a constant $\delta > 0$ dependent on $\frac{\xi}{2\|L\|_\infty}$ such that if $|U| \leq \delta n$, the LP decoder of G succeeds on $\frac{\tau}{\|L\|_\infty}$ and hence on τ . Thus, if

$$\frac{2\|L\|_\infty}{\xi(k-1)} \leq \delta,$$

then τ has a dual witness $v : E \rightarrow \mathbb{R}$ in G . Since $k \geq d$, let $v^k : \overline{E}^k \rightarrow \mathbb{R}$ be the extension of v to \overline{G}^k by zeros. Thus $\mathbf{F}(v^k) < \tau$ and accordingly

$$\mathbf{F}(w^k + v^k) < \gamma.$$

Therefore, $w^k + v^k$ is the desired dual witness of γ in \overline{G}^k . It follows (from Theorem 2.2 in [7]) that the \overline{G}^k -LP decoder successfully corrects γ .

In summary, there exists a constant $\delta > 0$ dependent on $\frac{\xi}{2\|L\|_\infty}$ such that if

$$k = \max \left\{ d, \left\lceil \frac{2\|L\|_\infty}{\xi\delta} \right\rceil + 1 \right\},$$

which depends on the ξ and the channel, then the \overline{G}^k -LP decoder corrects γ for any $\gamma \in \mathbb{R}^n$ such that the \overline{G} -LP decoder corrects $\gamma - \xi \mathbf{1}$. \square

Note that the proof of Lemma 3.2 breaks down if the LLRs are unbounded even if Lemma 2.2 holds for channels with unbounded LLRs (see Remark 2.3.II).

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